Generalizations of Right-a Fibonomial Numbers: The Double, Triple, and RF-Trinomial forms

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Abstract:
Since the start of the twentieth century, mathematicians have worked on extending the idea of binomial coefficients in many different directions. One well-known example is the Fibonomial coefficient, which is formed by replacing the usual natural numbers with Fibonacci numbers in the binomial formula. In recent past, this idea has been taken further with the introduction of a new variation called the Right a-Fibonomial Numbers. In this paper, we explore their special cases known as the Right-a Double Fibonomial Numbers, Right-a Triple Fibonomial Numbers and RF-Trinomial numbers showing when they result in integer values. Furthermore, we discuss several additional features, including their upper and lower bounds, which highlight the broader significance of Right-a Fibonomial Numbers in the study of generalized binomial coefficients.

1. Introduction:
In 2002, Kalman and Mena [9] explored the study related to the sequence of generalized Fibonacci numbers. These numbers are defined as follows:

Definition: For any positive integers a and b, the *generalized Fibonacci numbers* $\left\{F_n^{(a,b)}\right\}$ are defined recursively by the relation $F_n^{(a,b)} = aF_{n-1}^{(a,b)} + bF_{n-2}^{(a,b)}$; $n \geq 2$, where $F_0^{(a,b)} = 0. F_1^{(a,b)} = 1$.

First few terms of this sequence are $0, 1, a, a^2 + b, a^3 + 2ab, a^4 + 3a^2b + b^2, \dots$. It can be easily observed that $F_n^{(1,1)} = F_n$, the n^th^ Fibonacci number.

In 2015, Arvadia and Shah [1] studied the following sequence of right a-Fibonacci numbers, which is a special case of $\left\{F_n^{(1,a)}\right\}$.

Definition: For any positive integer a, the *right a-Fibonacci numbers* are defined by the recurrence relation $F_n^{(1,a)} = F_{n-1}^{(1,a)} + aF_{n-2}^{(1,a)}$; $n \geq 2$, where $F_0^{(1,a)} = 0$ and $F_1^{(1,a)} = 1$.

First few terms of this sequence are $0, 1, 1, 1 + a, 1 + 2a, 1 + 3a + a^2, \dots$. Arvadia and Shah [1] obtained extended Binet formula for this sequence as $F_n^{(1,a)} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \frac{1+\sqrt{1+4a}}{2}$ and $\beta = \frac{1-\sqrt{1+4a}}{2}$.

Moving parallel to this, we define the sequence of *right a-Lucas numbers* by the recurrence relation $L_n^{(1,a)} = L_{n-1}^{(1,a)} + aL_{n-2}^{(1,a)}$; $n \geq 2$, where $L_0^{(1,a)} = 2$ and $L_1^{(1,a)} = 1$. First few terms of this sequence are $2, 1, 1 + 2a, 1 + 3a, 1 + 4a + 2a^2, 1 + 5a + 5a^2, \dots$. Clearly, $L_n^{(1,1)} = L_n$, the n^th^ Lucas number. It easy to show that the extended Binet formula for this sequence of numbers is given by $L_n^{(1,a)} = \alpha^n + \beta^n$, where $\alpha = \frac{1+\sqrt{1+4a}}{2}$ and $\beta = \frac{1-\sqrt{1+4a}}{2}$.

In 2008, Benjamin and Plott [2] further defined the concept of Fibonomial numbers and derived interesting properties related to them. Subsequently, in 2018, Shah and Shah [15] introduced the Genomial numbers, replacing Fibonacci numbers with k-Fibonacci numbers through generalization. In the near past, Desai and Shah [5] unveiled the proper right a-Fibonomial numbers $\binom{m}{k}_{RF}$, delineated as $\binom{m}{k}_{RF} = \frac{F_m^{(1,a)*}}{F_k^{(1,a)*} \times F_{m-k}^{(1,a)*}}; 0 \le k \le m$, where $F_m^{(1,a)*} = F_m^{(1,a)} \times F_{m-1}^{(1,a)} \times F_{m-2}^{(1,a)} \times \cdots \times F_1^{(1,a)}$ and $F_0^{(1,a)*} = 1$ and then obtained properties and identities related to them.

In combinatorics, the factorial of a positive integer n, denoted by $n!$, is defined by $n! = n \times (n-1) \times \cdots \times 2 \times 1; n \ge 1$ and $0! = 1$. Whereas the double factorial of a positive integer n is usually denoted by $n!!$, and it is defined as follows:

Definition: For any non-negative integer n, double factorial is defined as

$$n!! = \begin{cases} n \times (n-2) \times \cdots \times 3 \times 1; n \text{ is odd} \\ n \times (n-2) \times \cdots \times 4 \times 2; n \text{ is even.} \\ 1 \qquad ; n = 0 \end{cases}$$

The double factorial binomial coefficients are defined as follows:

Definition: For any non-negative integers n and k, double factorial binomial coefficients are defined by

$$\left(\binom{n}{k}\right) = \frac{n!!}{k!! \times (n-k)!!}.$$

In 2021, Shah and Shah [14] introduced generalized double Fibonorial numbers and double Fibonomial numbers in which they replaced natural numbers by the terms of generalized Fibonacci numbers, $w_n = pw_{n-1} + qw_{n-2}$, for $n \ge 2$; $w_0 = a$ and $w_1 = b$, where a, b, p and q are any integers.

Definition: For any integer $n \ge 0$, generalized double Fibonorial numbers are defined by

$$n!!_w \equiv \begin{cases} 1 \qquad ; n = 0 \\ w_n \times w_{(n-2)} \times \cdots \times w_6 \times w_4 \times w_2 \,; n \text{ is even} \\ w_n \times w_{(n-2)} \times \cdots \times w_5 \times w_3 \times w_1 \,; n \text{ is odd} \end{cases}.$$

Definition: For $0 \le k \le n$, generalized double Fibonomial numbers are defined by

$$\left(\binom{n}{k}\right)_w = \frac{n!!_w}{k!!_w \times (n-k)!!_w}.$$

Also, the greatest common divisors of the binomial coefficients forming each of the two triangles in the Star of David shape in Pascal's triangle are equal. In 1971, Gould [7] obtained the interesting identity for binomial coefficients that is $\binom{n-a}{k-a}\binom{n}{k+a}\binom{n+a}{k} = \binom{n-a}{k}\binom{n+a}{k+a}\binom{n}{k-a}$.

In 2024, Shah and Shah [13] introduced an extension of Fibonomial numbers into three dimensions, termed F-trinomial numbers. In this extension, n is partitioned into three parts.

2. Right-a double Fibonomial Numbers:

Definition: For any integer $n \geq 0$, Right-a double Fibonorial numbers are defined by

$$n!!_{RF} \equiv \begin{cases} 1 & ;\ n = 0 \\ F_n^{R(1,a)} \times F_{n-2}^{R(1,a)} \times \cdots \times F_6^{R(1,a)} \times F_4^{R(1,a)} \times F_2^{R(1,a)} & ;\ n\ is\ even \\ F_n^{R(1,a)} \times F_{n-2}^{R(1,a)} \times \cdots \times F_5^{R(1,a)} \times F_3^{R(1,a)} \times F_1^{R(1,a)} & ;\ n\ is\ odd \end{cases}.$$

The definition of $n!!_{RF}$ helps to express the Right-a double Fibonorial in terms of regular Right-a Fibonorial as shown in the following lemma.

Lemma 2.1: $n!!_{RF} = \frac{n!_{RF}}{(n-1)!!_{RF}} = \frac{(n+1)!_{RF}}{(n+1)!!_{RF}}$; $n \geq 1$.

We use the concept of Right-a double Fibonorial to further generalize the concept of Right-a double Fibonomial coefficients defined as follows:

Definition: For $0 \leq k \leq n$, Right-a double Fibonomial numbers are defined by

$$\left(\binom{n}{k}\right)_{RF} = \frac{n!!_{RF}}{k!!_{RF} \times (n-k)!!_{RF}}.$$

It is easy to observe that $\left(\binom{n}{0}\right)_{RF} = 1 = \left(\binom{n}{n}\right)_{RF}, \left(\binom{n}{2}\right)_{RF} = bF_n^{R(a,b)}$ and $\left(\binom{n}{k}\right)_{RF} = \left(\binom{n}{n-k}\right)_{RF}$

2.1 Properties of Right-a double Fibonomial numbers:

The following results are easy consequences from the definition of Right-a double Fibonomial numbers.

Lemma 2.2: For any positive integers k, m and n,

1. Iterative Rule: $\left(\binom{n}{k}\right)_{RF}\left(\binom{k}{m}\right)_{RF} = \left(\binom{n}{m}\right)_{RF}\left(\binom{n-m}{k-m}\right)_{RF}$.
2. $F_{n-k}^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_n^{R(a,b)}\left(\binom{n-2}{k}\right)_{RF}$.
3. $F_{n-k}^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_{k+2}^{R(a,b)}\left(\binom{n}{k+2}\right)_{RF}$.
4. $F_k^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_{n-k+2}^{R(a,b)}\left(\binom{n}{k-2}\right)_{RF}$.

$$5.\, F_k^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_n^{R(a,b)}\left(\binom{n-2}{k-2}\right)_{RF}.$$

Proof: Using the definition of Right-a double Fibonomial numbers, we get

$$1.\left(\binom{n}{k}\right)_{RF}\left(\binom{k}{m}\right)_{RF} = \frac{n!!_{RF}}{k!!_{RF}\times(n-k)!!_{RF}}\times\frac{k!!_{RF}}{m!!_{RF}\times(k-m)!!_{RF}} = \frac{n!!_{RF}}{m!!_{RF}\times(n-m)!!_{RF}}\times$$

$$\frac{(n-m)!!_{RF}}{(k-m)!!_{RF}\times(n-k)!!_{RF}}.\text{ Thus, }\left(\binom{n}{k}\right)_{RF}\left(\binom{k}{m}\right)_{RF} = \left(\binom{n}{m}\right)_{RF}\left(\binom{n-m}{k-m}\right)_{RF}.$$

$$2.\, F_{n-k}^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_{n-k}^{R(a,b)}\times\frac{n!!_{RF}}{k!!_{RF}\times(n-k)!!_{RF}} = F_n^{R(a,b)}\times\frac{(n-2)!!_{RF}}{k!!_{RF}\times(n-k-2)!!_{RF}}.\text{ Thus,}$$

$$F_{n-k}^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_n^{R(a,b)}\left(\binom{n-2}{k}\right)_{RF}.$$

$$3.\, F_{n-k}^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_{n-k}^{R(a,b)}\times\frac{n!!_{RF}}{k!!_{RF}\times(n-k)!!_{RF}} = F_{k+2}^{R(a,b)}\times\frac{n!!_{RF}}{(k+2)!!_{RF}\times(n-k-2)!!_{RF}}.\text{ Thus,}$$

$$F_{n-k}^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_{k+2}^{R(a,b)}\left(\binom{n}{k+2}\right)_{RF}.$$

$$4.\, F_k^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_k^{R(a,b)}\times\frac{n!!_{RB}}{k!!_{RF}\times(n-k)!!_{RF}} = F_{n-k+2}^{R(a,b)}\times\frac{n!!_{RF}}{(k-2)!!_{RF}\times(n-k+2)!!_{RF}}.\text{ Thus,}$$

$$F_k^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_{n-k+2}^{R(a,b)}\left(\binom{n}{k-2}\right)_{RF}.$$

$$5.\, F_k^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_k^{R(a,b)}\times\frac{n!!_{RF}}{k!!_{RF}\times(n-k)!!_{RF}} = F_n^{R(a,b)}\times\frac{(n-2)!!_{RF}}{(k-2)!!_{RF}\times(n-k)!!_{RF}}.\text{ Thus,}$$

$$F_k^{R(a,b)}\left(\binom{n}{k}\right)_{RF} = F_n^{R(a,b)}\left(\binom{n-2}{k-2}\right)_{RF}.$$

Lemma 2.3: $(n-1)!!_{RF}\left(\binom{n}{k}\right)_{RF}$ is always an integer.

This result can be easily derived from the definition of Right-a Fibonorial and Right-a double Fibonomial numbers. The basic recurrence relations for the Right-a double Fibonomial numbers are as follows:

Lemma 2.4: $\left(\binom{n}{k}\right)_{RF} - \left(\binom{n-2}{k}\right)_{RF} = \left(\binom{n-2}{k-2}\right)_{RF}\left\{\frac{F_n^{R(a,b)} - F_{n-k}^{R(a,b)}}{F_k^{R(a,b)}}\right\}.$

By changing k to $n-k$ and using 2.3, we get

Lemma 2.5: $\left(\binom{n}{k}\right)_{RF} - \left(\binom{n-2}{k-2}\right)_{RF} = \left(\binom{n-2}{k}\right)_{RF}\left\{\frac{F_n^{R(a,b)} - F_k^{R(a,b)}}{F_{n-k}^{R(a,b)}}\right\}.$

The following result can be easily obtained when we apply the summation on both sides with respect to the upper index such that m and k have the same parity.

2.3 Star of David theorem for Right-a double Fibonomial numbers:

In 1972, Gould obtained a result related to one interesting arithmetic property of binomial coefficients which was named as the Star of David theorem. This theorem states that "The greatest common divisors of the binomial coefficients forming each of the two triangles in the Star of David shape in Pascal's triangle are equal". That is $\gcd\left\{\binom{n-1}{k-1},\binom{n}{k+1},\binom{n+1}{k}\right\} = \gcd\left\{\binom{n-1}{k},\binom{n}{k-1},\binom{n+1}{k+1}\right\}$.

The two sets of three numbers, which the Star of David theorem says, have equal greatest common divisors and equal products. Interestingly, Gould's result can be imitated for Right-a double Fibonomial numbers too as shown in the following result.

Theorem 2.6:

$$\left(\binom{n-p}{k-q}\right)_{RF}\left(\binom{n}{k+p}\right)_{RF}\left(\binom{n+q}{k}\right)_{RF} = \left(\binom{n-p}{k}\right)_{RF}\left(\binom{n+q}{k+p}\right)_{RF}\left(\binom{n}{k-q}\right)_{RF}\text{; where}$$

a, b are positive integers.

Proof: Using the definition of Right-a double Fibonomial numbers, the left side of the result becomes

$$\left(\binom{n-p}{k-q}\right)_{RF}\left(\binom{n}{k+p}\right)_{RF}\left(\binom{n+q}{k}\right)_{RF}$$

$$= \frac{(n-p)!!_{RF}}{(k-q)!!_{RF}\times(n-k-p+q)!!_{RF}} \times \frac{n!!_{RF}}{(k+p)!!_{RF}\times(n-k-p)!!_{RF}} \times \frac{(n+q)!!_{RF}}{k!!_{RF}\times(n-k+q)!!_{RF}}$$

$$= \frac{(n-p)!!_{RF}}{k!!_{RF}\times(n-k-p)!!_{RF}} \times \frac{(n+q)!!_{RF}}{(k+p)!!_{RF}\times(n-k-p+q)!!_{RF}} \times \frac{n!!_{RF}}{(k-q)!!_{RF}\times(n-k+q)!!_{RF}}$$

$$= \left(\binom{n-p}{k}\right)_{RF}\left(\binom{n+q}{k+p}\right)_{RF}\left(\binom{n}{k-q}\right)_{RF}\text{, as required.}$$

Note: If $a = b = 1$, we get the product of six Right-a double Fibonomial numbers, which are equally spaced around $\left(\binom{n}{k}\right)_{RF}$.

"Figure 1 is about here".

2.4 Right-a double multinomial numbers:

Right-a double multinomial number for any positive integer n is defined as follows:

Definition: For any integer $n > 0$, the Right-a double multinomial number is defined by

$$\left(\binom{n}{k_1, k_2, \cdots, k_r}\right)_{RF} = \frac{n!!_{RF}}{k_1!!_{RF}k_2!!_{RF}\cdots k_r!!_{RF}};\ n = k_1 + k_2 + \cdots + k_r.$$

Following result expresses Right-a double multinomial numbers as the product of Right-a bifurcating double Fibonomial numbers.

Lemma 2.7: Right-a double multinomial numbers can always be expressed as the product of Right-a double Fibonomial numbers.

Proof: We prove this result using the principle of mathematical induction on the value of r in the definition of Right-a double multinomial numbers.

When we consider $r = 2$, we have $\left(\binom{\mathrm{n}}{\mathrm{k}_1, \mathrm{k}_2}\right)_{\mathrm{RF}} = \left(\binom{\mathrm{n}}{\mathrm{k}_1}\right)_{\mathrm{RF}}$; where $k_1 + k_2 = n$.

For $r = 3$ and $n = k_1 + k_2 + k_3$,

$\left(\binom{\mathrm{n}}{\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3}\right)_{\mathrm{RF}} = \frac{n!!_{RF}}{k_1!!_{RF} k_2!!_{RF} k_3!!_{RF}} = \frac{n!!_{RF}}{k_1!!_{RF}(n-k_1)!!_{RF}} \times \frac{(n-k_1)!!_{RF}}{k_2!!_{RF} k_3!!_{RF}}$. As $k_3 = n - k_1 - k_2$, we get

$$\left(\binom{\mathrm{n}}{\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3}\right)_{\mathrm{RF}} = \left(\binom{\mathrm{n}}{\mathrm{k}_1}\right)_{\mathrm{RF}} \left(\binom{\mathrm{n} - \mathrm{k}_1}{\mathrm{k}_2}\right)_{\mathrm{RF}}.$$

Let for $r \leq m - 1$ and $n = k_1 + k_2 + \cdots + k_r$, we have

$$\left(\binom{\mathrm{n}}{\mathrm{k}_1, \mathrm{k}_2, \cdots, \mathrm{k}_\mathrm{r}}\right)_{\mathrm{RF}} = \left(\binom{\mathrm{n}}{\mathrm{k}_1}\right)_{\mathrm{RF}} \left(\binom{\mathrm{n} - \mathrm{k}_1}{\mathrm{k}_2}\right)_{\mathrm{RF}} \cdots \left(\binom{\mathrm{n} - \mathrm{k}_1 - \mathrm{k}_2 \cdots - \mathrm{k}_{\mathrm{r}-2}}{\mathrm{k}_{\mathrm{r}-1}}\right)_{\mathrm{RF}}.$$

Let us now consider $r = m$ and $n = k_1 + k_2 + \cdots + k_m$. Thus,

$$\left(\binom{\mathrm{n}}{\mathrm{k}_1, \mathrm{k}_2, \cdots, \mathrm{k}_\mathrm{r}}\right)_{\mathrm{RF}} = \frac{n!!_{RF}}{k_1!!_{RF}\, k_2!!_{RF} \cdots k_m!!_{RF}}$$

$$= \frac{n!!_{RF}}{k_1!!_{RF}\, k_2!!_{RF} \cdots k_{m-2}!!_{RF}} \times \frac{1}{k_{m-1}!!_{RF}\, k_m!!_{RF}}$$

$$= \frac{n!!_{RF}}{k_1!!_{RF} k_2!!_{RF} \cdots k_{m-2}!!_{RF} \times (n-k_1-k_2-\cdots-k_{m-2})!!_{RF}} \times \frac{(n-k_1-k_2-\cdots-k_{m-2})!!_{RF}}{k_{m-1}!!_{RF}(n-k_1-k_2-\cdots-k_{m-2}-k_{m-1})!!_{RF}}$$

$$= \left(\binom{\mathrm{n}}{k_1}\right)_{\mathrm{RF}} \left(\binom{\mathrm{n} - \mathrm{k}_1}{k_2}\right)_{\mathrm{RF}} \cdots \left(\binom{n - k_1 - k_2 - \cdots - k_{n-3}}{k_{n-2}}\right)_{\mathrm{RF}} \left(\binom{n - k_1 - k_2 - \cdots - k_{n-2}}{k_{n-1}}\right)_{\mathrm{RF}}.$$

Hence, by the principle of mathematical induction, we get the required result.

3. Right-*a* triple Fibonomial Numbers:

We introduce Right-a triple Fibonorial numbers which is the product of every third Fibonacci-type number up to n, beginning with $F_n^{(1,a)}$ and ending with $F_3^{(1,a)}$, $F_2^{(1,a)}$, or $F_1^{(1,a)}$ depending on whether n leaves a remainder of 0, 2, or 1 when divided by 3.

Definition: For any integer n ≥ 0, Right-a triple Fibonorial numbers are defined by

$$n!!!_{RF} \equiv \begin{cases} 1 & ; n = 0 \\ F_n^{(1,a)} \times F_{n-3}^{(1,a)} \times \cdots \times F_6^{(1,a)} \times F_3^{(1,a)}; & n \equiv 0 (mod\ 3) \\ F_n^{(1,a)} \times F_{n-3}^{(1,a)} \times \cdots \times F_4^{(1,a)} \times F_1^{(1,a)}; & n \equiv 1 (mod\ 3) \\ F_n^{(1,a)} \times F_{n-3}^{(1,a)} \times \cdots \times F_5^{(1,a)} \times F_2^{(1,a)}; & n \equiv 2 (mod\ 3) \end{cases}.$$

The definition of $n!!!_{RF}$ allows the right-a triple Fibonorial to be expressed in terms of the right-a double Fibonorial, as demonstrated in the following lemma.

Lemma 3.1: $n!!!_{RF} = \frac{n!!_{RF}(n-3)!!!_{RF}}{(n-2)!!_{RF}} = \frac{(n+3)!!!_{RF}}{F_{n+3}^{(1,a)}}$; $n \geq 3$.

We use above identity to easily show the relation between $n!_{RF}, n!!_{RF}$ and $n!!!_{RF}$ by use of their definitions.

Lemma 3.2: $\boldsymbol{n!!!_{RF} = \frac{n!_{RF}(n-3)!!!_{RF}}{(n-1)!!_{RF}(n-2)!!_{RF}};\ n \geq 3.}$

We also have the relation between regular right-a Fibonorial numbers and right-a triple Fibonomial numbers.

Lemma 3.3: $\boldsymbol{n!!!_{RF} = \frac{n!_{RF}(n-3)!!!_{RF}}{(n-1)!_{RF}};\ n \geq 3.}$

Next, we define the concept of right-a triple Fibonomial coefficients in terms of right-a triple Fibonorial numbers:

Definition: *For* $0 \leq k \leq n$, *Right-a triple Fibonomial numbers* are defined by

$$\left(\left(\binom{n}{k}\right)\right)_{RF} = \frac{n!!_{RF}}{k!!_{RF} \times (n-k)!!_{RF}}.$$

It is easy to observe that $\left(\left(\binom{n}{0}\right)\right)_{RF} = 1 = \left(\left(\binom{n}{n}\right)\right)_{RF}$ and $\left(\left(\binom{n}{k}\right)\right)_{RF} = \left(\left(\binom{n}{n-k}\right)\right)_{RF}$.

3.1 Properties of Right-a triple Fibonomial numbers:

The results presented below are immediate consequences of the definition of the right-a triple Fibonomial numbers.

Lemma 3.4: For any positive integers k, m and n,

1. Iterative Rule: $\left(\left(\binom{n}{k}\right)\right)_{RF}\left(\left(\binom{k}{m}\right)\right)_{RF} = \left(\left(\binom{n}{m}\right)\right)_{RF}\left(\left(\binom{n-m}{k-m}\right)\right)_{RF}$.

2. $F_{n-k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n}^{(1,a)}\left(\left(\binom{n-3}{k}\right)\right)_{RF}$.

3. $F_{n-k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{k}^{(1,a)}\left(\left(\binom{n}{k+3}\right)\right)_{RF}$.

4. $F_{k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n-k+3}^{(1,a)}\left(\left(\binom{n}{k-3}\right)\right)_{RF}$.

5. $F_{k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n}^{(1,a)}\left(\left(\binom{n-3}{k-3}\right)\right)_{RF}$.

Proof: Using the definition of right-*a* triple Fibonomial numbers, we get

1. $\left(\left(\binom{n}{k}\right)\right)_{RF}\left(\left(\binom{k}{m}\right)\right)_{RF} = \frac{n!!!_{RF}}{k!!!_{RF}\times(n-k)!!!_{RF}}\times\frac{k!!!_{RF}}{m!!!_{RF}\times(k-m)!!!_{RF}} = \frac{n!!!_{RF}}{m!!!_{RF}\times(n-m)!!!_{RF}}\times\frac{(n-m)!!!_{RF}}{(k-m)!!!_{RF}\times(n-k)!!!_{RF}}$. Thus, $\left(\left(\binom{n}{k}\right)\right)_{RF}\left(\left(\binom{k}{m}\right)\right)_{RF} = \left(\left(\binom{n}{m}\right)\right)_{RF}\left(\left(\binom{n-m}{k-m}\right)\right)_{RF}$.

2. $F_{n-k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n-k}^{(1,a)}\times\frac{n!!!_{RF}}{k!!!_{RF}\times(n-k)!!!_{RF}} = F_{n}^{(1,a)}\times\frac{(n-3)!!!_{RF}}{k!!!_{RF}\times(n-k-3)!!!_{RF}}$.

Thus, $F_{n-k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n}^{(1,a)}\left(\left(\binom{n-3}{k}\right)\right)_{RF}$.

3. $F_{n-k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n-k}^{(1,a)}\times\frac{n!!!_{RF}}{k!!!_{RF}\times(n-k)!!!_{RF}} = F_{k}^{(1,a)}\times\frac{n!!!_{RF}}{(k+3)!!_{RF}\times(n-k-3)!!_{RF}}$.

Thus, $F_{n-k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{k}^{(1,a)}\left(\left(\binom{n}{k+3}\right)\right)_{RF}$.

4. $F_{k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{k}^{(1,a)}\times\frac{n!!!_{w}}{k!!!_{w}\times(n-k)!!!_{w}} = F_{n-k+3}^{(1,a)}\times\frac{n!!!_{w}}{(k-3)!!!_{w}\times(n-k+3)!!!_{w}}$.

Thus, $F_{k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n-k+3}^{(1,a)}\left(\left(\binom{n}{k-3}\right)\right)_{RF}$.

5. $F_{k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{k}^{(1,a)}\times\frac{n!!!_{RF}}{k!!!_{RF}\times(n-k)!!!_{RF}} = F_{n}^{(1,a)}\times\frac{(n-3)!!!_{RF}}{(k-3)!!!_{RF}\times(n-k)!!!_{RF}}$.

Thus, $F_{k}^{(1,a)}\left(\left(\binom{n}{k}\right)\right)_{RF} = F_{n}^{(1,a)}\left(\left(\binom{n-3}{k-3}\right)\right)_{RF}$.

The recurrence relations for the right-*a* triple Fibonomial numbers are as follows:

Lemma 3.5: $\left(\left(\binom{n}{k}\right)\right)_{RF} - \left(\left(\binom{n-3}{k}\right)\right)_{RF} = \left(\left(\binom{n-3}{k-3}\right)\right)_{RF}\left\{\frac{F_{n}^{(1,a)}-F_{n-k}^{(1,a)}}{F_{k}^{(1,a)}}\right\}$.

By changing k to n − k, we get

Lemma 3.6: $\left(\left(\binom{n}{k}\right)\right)_{RF} - \left(\left(\binom{n-3}{k-3}\right)\right)_{RF} = \left(\left(\binom{n-3}{k}\right)\right)_{RF}\left\{\frac{F_{n}^{(1,a)}-F_{k}^{(1,a)}}{F_{n-k}^{(1,a)}}\right\}$.

3.2 Star of David theorem for Right-a double Fibonomial numbers:

In 1972, Gould established a result concerning an intriguing arithmetic property of binomial coefficients, known as the *Star of David Theorem*. This theorem asserts that the greatest common divisors of the binomial coefficients forming each of the two interlaced triangles in the Star of David pattern within Pascal's triangle are equal. That is

$$gcd\left\{\binom{n-1}{k-1},\binom{n}{k+1},\binom{n+1}{k}\right\} = gcd\left\{\binom{n-1}{k},\binom{n}{k-1},\binom{n+1}{k+1}\right\}.$$

According to the Star of David Theorem, the two corresponding sets of three numbers share identical greatest common divisors and products. Remarkably, an analogous property also holds for the right-a triple Fibonomial numbers, as established in the next result.

Theorem 3.7: $\left(\left(\binom{n-p}{k-q}\right)\right)_{RF}\left(\left(\binom{n}{k+p}\right)\right)_{RF}\left(\left(\binom{n+q}{k}\right)\right)_{RF} =$

$\left(\left(\binom{n-p}{k}\right)\right)_{RF}\left(\left(\binom{n}{k-q}\right)\right)_{RF}\left(\left(\binom{n+q}{k+p}\right)\right)_{RF}$; where a, b are positive integers.

Proof: Using the definition of right-*a* triple Fibonomial numbers, the left side of the result becomes

$$\left(\left(\binom{n-p}{k-q}\right)\right)_{RF}\left(\left(\binom{n}{k+p}\right)\right)_{RF}\left(\left(\binom{n+q}{k}\right)\right)_{RF}$$

$$= \frac{(n-p)!!!_{RF}}{(k-q)!!!_{RF}\times(n-k-p+q)!!!_{RF}} \times \frac{n!!!_{RF}}{(k+p)!!!_{RF}\times(n-k-p)!!!_{RF}} \times \frac{(n+q)!!!_{RF}}{k!!!_{RF}\times(n-k+q)!!!_{RF}}$$

$$= \frac{(n-p)!!!_{RF}}{k!!!_{RF}\times(n-k-p)!!!_{RF}} \times \frac{n!!!_{RF}}{(k-q)!!!_{RF}\times(n-k+q)!!!_{RF}} \times \frac{(n+q)!!!_{RF}}{(k+p)!!!_{RF}\times(n-k-p+q)!!!_{RF}}$$

$$= \left(\left(\binom{n-p}{k}\right)\right)_{RF}\left(\left(\binom{n}{k-q}\right)\right)_{RF}\left(\left(\binom{n+q}{k+p}\right)\right)_{RF}$$, as required.

Note: If $a = b = 1$, we get the product of six right-*a* triple Fibonomial numbers, which are equally spaced around $\left(\left(\binom{n}{k}\right)\right)_{RF}$.

"Figure 2 is about here".

3.4 Right-a triple multinomial numbers:

Right-a triple multinomial number for any positive integer n is defined as follows:

Definition: For any integer $n > 0$, the right-a triple multinomial number is defined by

$$\left(\left(\binom{n}{k_1, k_2, \cdots, k_r}\right)\right)_{RF} = \frac{n!!!_{RF}}{k_1!!!_{RF}k_2!!!_{RF}\cdots k_r!!!_{RF}}; n = k_1 + k_2 + \cdots + k_r.$$

Following result expresses right-a triple multinomial numbers as the product of right-a triple Fibonomial numbers.

Lemma 3.8: Right-*a* triple multinomial numbers can always be expressed as the product of right-*a* triple Fibonomial numbers.

Proof: We prove this result using the principle of mathematical induction on the value of r in the definition of right-*a* triple multinomial number.

When we consider $r = 2$, we have $\left(\left(\binom{n}{k_1, k_2}\right)\right)_{RF} = \left(\left(\binom{n}{k_1}\right)\right)_{RF}$; where $k_1 + k_2 = n$.

For $r = 3$ and $n = k_1 + k_2 + k_3$,

$\left(\left(\left(\binom{n}{k_1, k_2, k_3}\right)\right)\right)_{RF} = \frac{n!!!_{RF}}{k_1!!!_{RF}k_2!!!_{RF}k_3!!!_{RF}} = \frac{n!!!_{RF}}{k_1!!!_{RF}(n-k_1)!!!_{RF}} \times \frac{(n-k_1)!!!_{RF}}{k_2!!!_{RF}k_3!!!_{RF}}$. As $k_3 = n - k_1 - k_2$, we get

$$\left(\left(\left(\binom{n}{k_1, k_2, k_3}\right)\right)\right)_{RF} = \left(\left(\left(\binom{n}{k_1}\right)\right)\right)_{RF}\left(\left(\left(\binom{n-k_1}{k_2}\right)\right)\right)_{RF}.$$

Let for $r \leq m - 1$ *and* $n = k_1 + k_2 + \cdots + k_r$, we have

$$\left(\left(\binom{n}{k_1, k_2, \cdots, k_r}\right)\right)_{RF} = \left(\binom{n}{k_1}\right)_{RF}\left(\binom{n-k_1}{k_2}\right)_{RF} \cdots \left(\binom{n-k_1-k_2\cdots-k_{r-2}}{k_{r-1}}\right)_{RF}.$$

Let us now consider $r = m$ and $n = k_1 + k_2 + \cdots + k_m$. Thus,

$$\left(\left(\left(\binom{n}{k_1, k_2, \cdots, k_r}\right)\right)\right)_{RF} = \frac{n!!!_{RF}}{k_1!!!_{RF}\, k_2!!!_{RF} \cdots k_m!!!_{RF}}$$

$$= \frac{n!!!_{RF}}{k_1!!!_{RF}\, k_2!!!_{RF} \cdots k_{m-2}!!!_{RF}} \times \frac{1}{k_{m-1}!!!_{RF}\, k_m!!!_{RF}}$$

$$= \frac{n!!!_{RF}}{k_1!!!_{RF}k_2!!!_{RF}\cdots k_{m-2}!!!_{RF}\times(n-k_1-k_2-\cdots-k_{m-2})!!!_{RF}} \times \frac{(n-k_1-k_2-\cdots-k_{m-2})!!!_{RF}}{k_{m-1}!!!_{RF}(n-k_1-k_2-\cdots-k_{m-2}-k_{m-1})!!!_{RF}}$$

$$= \left(\left(\left(\binom{n}{k_1}\right)\right)\right)_{RF}\left(\left(\left(\binom{n-k_1}{k_2}\right)\right)\right)_{RF} \cdots \left(\left(\left(\binom{n-k_1-k_2-\cdots-k_{n-3}}{k_{n-2}}\right)\right)\right)_{RF}\left(\left(\left(\binom{n-k_1-k_2-\cdots-k_{n-2}}{k_{n-1}}\right)\right)\right)_{RF}$$

Hence, by the principle of mathematical induction, we get the required result.

4. RF-Trinomial numbers:

The two-dimensional generalization of Fibonomial numbers has been extensively studied in the literature. In these generalizations, the integer n is typically divided into two parts. Later, Shah and Shah [13] introduced the concept of F-trinomial numbers as a three-dimensional extension by partitioning n into three parts. In this study we introduce *RF-trinomial numbers* for right-a Fibonacci numbers:

Definition: For any positive integer n and non-negative integers r, s, t such that $r + s + t = n$ and

$F_n^{(1,a)^*} = F_n^{(1,a)} \times F_{n-1}^{(1,a)} \times \cdots \times F_1^{(1,a)}$, the *RF-trinomial numbers* are defined as

$\begin{bmatrix} n \\ r, s, t \end{bmatrix}_{RF} = \frac{F_n^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}}$.

Using this definition, we can easily obtain the following results.

4.1 Properties of RF-Trinomial numbers:

The following results are trivial from the definition of RF-trinomial numbers.

Lemma 4.1: $\begin{bmatrix} n \\ 0, s, t \end{bmatrix}_{RF} = \binom{n}{s}_{RF}$, the regular right-$a$ Fibonomial coefficient.

Lemma 4.2: $\begin{bmatrix} n \\ r, 0,0 \end{bmatrix}_{RF} = 1$.

Lemma 4.3: $\begin{bmatrix} n \\ 1, s, t \end{bmatrix}_{RF} = F_n^{(1,a)} \binom{n-1}{s}_{RF}$.

We can observe that in this case, RF-trinomial number is a product of a right-a Fibonacci number and right-a Fibonomial coefficient.

Lemma 4.4: $\begin{bmatrix} 2n \\ n, n, 0 \end{bmatrix}_{RF} = \begin{bmatrix} 2n-1 \\ n-1, n, 0 \end{bmatrix}_{RF} \times L_n^{(1,a)}$.

Proof: From the definition of RF-trinomial numbers, since $F_{2n}^{(1,a)} = F_n^{(1,a)} L_n^{(1,a)}$, we have

$$\begin{bmatrix} 2n \\ n, n, 0 \end{bmatrix}_{RF} = \frac{F_{2n}^{(1,a)^*}}{F_n^{(1,a)^*} F_n^{(1,a)^*} F_0^{(1,a)^*}} = \frac{F_{2n}^{(1,a)}}{F_n^{(1,a)}} \times \frac{F_{2n-1}^{(1,a)^*}}{F_{n-1}^{(1,a)^*} F_n^{(1,a)^*} F_0^{(1,a)^*}} = L_n^{(1,a)} \times \begin{bmatrix} 2n-1 \\ n-1, n, 0 \end{bmatrix}_{RF}.$$

We can observe that in this case, $\begin{bmatrix} 2n \\ n, n, 0 \end{bmatrix}_{RF}$ is always divisible by $L_n^{(1,a)}$.

The following result gives the recurrence relation connecting the F-trinomial numbers.

Lemma 4.5:

$$\begin{bmatrix} n \\ r, s, t \end{bmatrix}_{RF} = F_{s+t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r-1, s, t \end{bmatrix}_{RF} + aF_{r-1}^{(1,a)} F_{t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r, s-1, t \end{bmatrix}_{RF} + a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} \begin{bmatrix} n-1 \\ r, s, t-1 \end{bmatrix}_{RF}.$$

Proof: We know that for any right-a Fibonacci number $F_n^{(1,a)}$,

$F_{r+s}^{(1,a)} = F_r^{(1,a)} F_{s+1}^{(1,a)} + aF_{r-1}^{(1,a)} F_s^{(1,a)}$.

Using this result, we get $F_{r+s+t}^{(1,a)} = F_r^{(1,a)} F_{s+t+1}^{(1,a)} + aF_{r-1}^{(1,a)} F_{s+t}^{(1,a)}$

$$= F_r^{(1,a)} F_{s+t+1}^{(1,a)} + aF_{r-1}^{(1,a)} \left\{ F_s^{(1,a)} F_{t+1}^{(1,a)} + aF_{s-1}^{(1,a)} F_t^{(1,a)} \right\}.$$

Now as $n = r + s + t$, we get $F_n^{(1,a)^*} = F_r^{(1,a)} F_{s+t+1}^{(1,a)} + aF_{r-1}^{(1,a)} F_s^{(1,a)} F_{t+1}^{(1,a)} + a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} F_t^{(1,a)}$. Multiplying by $\frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}}$, we get

$$F_n^{(1,a)^*} \frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}}$$

$$= F_r^{(1,a)} F_{s+t+1}^{(1,a)} \frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}} + aF_{r-1}^{(1,a)} F_s^{(1,a)} F_{t+1}^{(1,a)} \frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}}$$

$$+ a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} F_t^{(1,a)} \frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}}.$$

Using the definition of RF-trinomial number, we have

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = F_{s+t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_{RF} + aF_{r-1}^{(1,a)} F_{t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF} +$$

$$a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s,t-1 \end{bmatrix}_{RF},$$

as desired.

This result can be illustrated by the following example.

Illustration 4.6: Consider $n = 6$ and $r = s = r = 2$.

Then we get $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = \begin{bmatrix} 6 \\ 2,2,2 \end{bmatrix}_{RF} = (1 + 4a + 3a^2)(1 + 3a + a^2)(1 + 2a)(1 + a)$.

Also, $\begin{bmatrix} 5 \\ 1,2,2 \end{bmatrix}_{RF} = \begin{bmatrix} 5 \\ 2,1,2 \end{bmatrix}_{RF} = \begin{bmatrix} 5 \\ 2,2,1 \end{bmatrix}_{RF} = (1 + 3a + a^2)(1 + 2a)(1 + a)$. Therefore,

$$F_{s+t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_{RF} + aF_{r-1}^{(1,a)} F_{t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF} + a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s,t-1 \end{bmatrix}_{RF}$$

$$= F_5^{(1,a)} \begin{bmatrix} 5 \\ 1,2,2 \end{bmatrix}_{RF} + aF_1^{(1,a)} F_3^{(1,a)} \begin{bmatrix} 5 \\ 2,1,2 \end{bmatrix}_{RF} + a^2 F_1^{(1,a)} F_1^{(1,a)} \begin{bmatrix} 5 \\ 2,2,1 \end{bmatrix}_{RF}$$

$$= (1 + 3a + a^2)(1 + 2a)(1 + a)\{1 + 3a + a^2 + a(1 + a) + a^2\}$$

$$= (1 + 3a + a^2)(1 + 2a)(1 + a)(1 + 4a + 3a^2)$$

Thus, $\begin{bmatrix} 6 \\ 2,2,2 \end{bmatrix}_{RF} = F_5^{(1,a)} \begin{bmatrix} 5 \\ 1,2,2 \end{bmatrix}_{RF} + aF_1^{(1,a)} F_3^{(1,a)} \begin{bmatrix} 5 \\ 2,1,2 \end{bmatrix}_{RF} + a^2 F_1^{(1,a)} F_1^{(1,a)} \begin{bmatrix} 5 \\ 2,2,1 \end{bmatrix}_{RF}$, as expected.

Following result uses the above lemma to give recurrence relation of RF-trinomial numbers in the series form.

Corollary 4.7:

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = \sum_{j=1}^{t} \left\{ \begin{matrix} a^{j-1} F_{r-1}^{(1,a)^{j-1}} F_{s-1}^{(1,a)^{j-1}} F_{s+t-j+2}^{(1,a)} \begin{bmatrix} n-j \\ r-1,s,t-j+1 \end{bmatrix}_{RF} \\ +a^{j} F_{r-1}^{(1,a)^{j}} F_{s-1}^{(1,a)^{j-1}} F_{t-j+2}^{(1,a)} \begin{bmatrix} n-j \\ r,s-1,t-j+1 \end{bmatrix}_{RF} \end{matrix} \right\} +$$

$$a^{2t} F_{r-1}^{(1,a)^t} F_{s-1}^{(1,a)^t} \begin{bmatrix} n-t \\ r,s,0 \end{bmatrix}_{RF}.$$

Proof: From lemma, we have

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = F_{s+t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_{RF} + aF_{r-1}^{(1,a)} F_{t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF} +$$

$$a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s,t-1 \end{bmatrix}_{RF}.$$

Applying the same result to the last term, we get $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF}$

$$= F_{s+t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_{RF} + aF_{r-1}^{(1,a)} F_{t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF} +$$

$$a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} \left\{ \begin{matrix} F_{s+t}^{(1,a)} \begin{bmatrix} n-2 \\ r-1,s,t-1 \end{bmatrix}_{RF} \\ +aF_{r-1}^{(1,a)} F_{t}^{(1,a)} \begin{bmatrix} n-2 \\ r,s-1,t-1 \end{bmatrix}_{RF} + a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} \begin{bmatrix} n-2 \\ r,s,t-2 \end{bmatrix}_{RF} \end{matrix} \right\}$$

$$= F_{s+t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_{RF} + a^2 F_{r-1}^{(1,a)} F_{s-1}^{(1,a)} F_{s+t}^{(1,a)} \begin{bmatrix} n-2 \\ r-1,s,t-1 \end{bmatrix}_{RF}$$

$$+a^4 F_{r-1}^{(1,a)^2} F_{s-1}^{(1,a)^2} F_{s+t-1}^{(1,a)} \begin{bmatrix} n-3 \\ r-1,s,t-2 \end{bmatrix}_{RF} + aF_{r-1}^{(1,a)} F_{t+1}^{(1,a)} \begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF}$$

$$+a^3 F_{r-1}^{(1,a)^2} F_{s-1}^{(1,a)} F_{t}^{(1,a)} \begin{bmatrix} n-2 \\ r,s-1,t-1 \end{bmatrix}_{RF} + a^5 F_{r-1}^{(1,a)^3} F_{s-1}^{(1,a)^2} F_{t-1}^{(1,a)} \begin{bmatrix} n-3 \\ r,s-1,t-2 \end{bmatrix}_{RF}$$

$$+a^6 F_{r-1}^{(1,a)^3} F_{s-1}^{(1,a)^3} \begin{bmatrix} n-3 \\ r,s,t-3 \end{bmatrix}_{RF}.$$

Continuing the process, we get

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = \sum_{j=1}^{t} \left\{ \begin{matrix} a^{j-1} F_{r-1}^{(1,a)^{j-1}} F_{s-1}^{(1,a)^{j-1}} F_{s+t-j+2}^{(1,a)} \begin{bmatrix} n-j \\ r-1,s,t-j+1 \end{bmatrix}_{RF} \\ +a^{j} F_{r-1}^{(1,a)^{j}} F_{s-1}^{(1,a)^{j-1}} F_{t-j+2}^{(1,a)} \begin{bmatrix} n-j \\ r,s-1,t-j+1 \end{bmatrix}_{RF} \end{matrix} \right\} +$$

$$a^{2t} F_{r-1}^{(1,a)^t} F_{s-1}^{(1,a)^t} \begin{bmatrix} n-t \\ r,s,0 \end{bmatrix}_{RF},$$

as required.

The above result can be illustrated by the following example.

Illustration 4.8: Consider $n = 5$ and $r = s = 2, t = 1$.

Then we have $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = \begin{bmatrix} 5 \\ 2,2,1 \end{bmatrix}_{RF} = (1 + 3a + a^2)(1 + 2a)(1 + a)$.

Also,
$$\sum_{j=1}^{t} \left\{ \begin{matrix} a^{j-1} F_{r-1}^{(1,a)^{j-1}} F_{s-1}^{(1,a)^{j-1}} F_{s+t-j+2}^{(1,a)} \begin{bmatrix} n-j \\ r-1,s,t-j+1 \end{bmatrix}_{RF} \\ +a^{j} F_{r-1}^{(1,a)^{j}} F_{s-1}^{(1,a)^{j-1}} F_{t-j+2}^{(1,a)} \begin{bmatrix} n-j \\ r,s-1,t-j+1 \end{bmatrix}_{RF} \end{matrix} \right\} +$$

$$a^{2t} F_{r-1}^{(1,a)^t} F_{s-1}^{(1,a)^t} \begin{bmatrix} n-t \\ r,s,0 \end{bmatrix}_{RF}$$

$$= \sum_{j=1}^{1} \left\{ \begin{matrix} a^0 F_1^{(1,a)^0} F_1^{(1,a)^0} F_{2+1-1+2}^{(1,a)} \begin{bmatrix} 4 \\ 1,2,1 \end{bmatrix}_{RF} \\ +a^1 F_1^{(1,a)^1} F_1^{(1,a)^0} F_2^{(1,a)} \begin{bmatrix} 4 \\ 2,1,1 \end{bmatrix}_{RF} \end{matrix} \right\} + a^2 F_1^{(1,a)^1} F_1^{(1,a)^1} \begin{bmatrix} 4 \\ 2,2,0 \end{bmatrix}_{RF}$$

$$= F_4^{(1,a)} \begin{bmatrix} 4 \\ 1,2,1 \end{bmatrix}_{RF} + aF_2^{(1,a)} \begin{bmatrix} 4 \\ 2,1,1 \end{bmatrix}_{RF} + a^2 \begin{bmatrix} 4 \\ 2,2,0 \end{bmatrix}_{F}$$

$= (1 + 2a)(1 + a)\{1 + 2a + a + a^2\}$

$= (1 + 2a)(1 + a)(1 + 3a + a^2)$, as required.

It is not evident from the definition of RF-trinomial numbers that they always possess integer values or not. In the following theorem, we show that they indeed always possess integer values.

Theorem 4.9: RF-trinomial numbers are always integers.

Proof: From the definition of RF-trinomial numbers, we have

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = \frac{F_n^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}} = \frac{\overbrace{F_n^{(1,a)} \times F_{n-1}^{(1,a)} \times \cdots \times F_{s+t+1}^{(1,a)}} \times \overbrace{F_{s+t}^{(1,a)} \times \cdots \times F_{t+1}^{(1,a)}} \times \overbrace{F_t^{(1,a)} \times \cdots \times F_1^{(1,a)}}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}}.$$

This fraction contains r, s and t number of consecutive right-a Fibonacci numbers in the numerator as well as in denominator. Since multiplication of any $'m'$ consecutive right-a Fibonacci numbers is always divisible by the multiplication of first $'m'$ consecutive right-a Fibonacci numbers [20], it is now evident that $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF}$ is always integer.

In [7], Gould gave an interesting result known as *Star of David theorem* for binomial coefficients, stated as $\binom{n-a}{r-a}\binom{n}{r+a}\binom{n+a}{r} = \binom{n-a}{r}\binom{n+a}{r+a}\binom{n}{r-a}$. In case of RF-trinomial numbers too we can discover similar result, which will be the 3-dimensional version of star of David theorem.

Theorem 4.10: $\begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF} \begin{bmatrix} n \\ r-1,s,t+1 \end{bmatrix}_{RF} \begin{bmatrix} n \\ r+1,s,t-1 \end{bmatrix}_{RF} \begin{bmatrix} n+1 \\ r,s+1,t \end{bmatrix}_{RF}$

$$= \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_{RF} \begin{bmatrix} n \\ r,s-1,t+1 \end{bmatrix}_{RF} \begin{bmatrix} n \\ r,s+1,t-1 \end{bmatrix}_{RF} \begin{bmatrix} n+1 \\ r+1,s,t \end{bmatrix}_{RF}$$

$$= \begin{bmatrix} n-1 \\ r,s,t-1 \end{bmatrix}_{RF} \begin{bmatrix} n \\ r-1,s+1,t \end{bmatrix}_{RF} \begin{bmatrix} n \\ r+1,s-1,t \end{bmatrix}_{RF} \begin{bmatrix} n+1 \\ r,s,t+1 \end{bmatrix}_{RF}.$$

Proof: Using the definition of RF-trinomial numbers, we have

$$\begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF} \begin{bmatrix} n \\ r-1,s,t+1 \end{bmatrix}_{RF} \begin{bmatrix} n \\ r+1,s,t-1 \end{bmatrix}_{RF} \begin{bmatrix} n+1 \\ r,s+1,t \end{bmatrix}_{RF}$$

$$= \frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_{s-1}^{(1,a)^*} F_t^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_{r-1}^{(1,a)^*} F_s^{(1,a)^*} F_{t+1}^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_{r+1}^{(1,a)^*} F_s^{(1,a)^*} F_{t-1}^{(1,a)^*}} \times \frac{F_{n+1}^{(1,a)^*}}{F_r^{(1,a)^*} F_{s+1}^{(1,a)^*} F_t^{(1,a)^*}}$$

$$= \frac{F_{n-1}^{(1,a)^*}}{F_{r-1}^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_r^{(1,a)^*} F_{s-1}^{(1,a)^*} F_{t+1}^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_r^{(1,a)^*} F_{s+1}^{(1,a)^*} F_{t-1}^{(1,a)^*}} \times \frac{F_{n+1}^{(1,a)^*}}{F_{r+1}^{(1,a)^*} F_s^{(1,a)^*} F_t^{(1,a)^*}}$$

$$= \begin{bmatrix} n-1 \\ r-1,s,t \end{bmatrix}_F \begin{bmatrix} n \\ r,s-1,t+1 \end{bmatrix}_F \begin{bmatrix} n \\ r,s+1,t-1 \end{bmatrix}_F \begin{bmatrix} n+1 \\ r+1,s,t \end{bmatrix}_F.$$

Again, using the definition of RF-trinomial numbers, we have

$$\begin{bmatrix} n-1 \\ r,s-1,t \end{bmatrix}_{RF} \begin{bmatrix} n \\ r-1,s,t+1 \end{bmatrix}_{RF} \begin{bmatrix} n \\ r+1,s,t-1 \end{bmatrix}_{RF} \begin{bmatrix} n+1 \\ r,s+1,t \end{bmatrix}_{RF}$$

$$= \frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_{s-1}^{(1,a)^*} F_t^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_{r-1}^{(1,a)^*} F_s^{(1,a)^*} F_{t+1}^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_{r+1}^{(1,a)^*} F_s^{(1,a)^*} F_{t-1}^{(1,a)^*}} \times \frac{F_{n+1}^{(1,a)^*}}{F_r^{(1,a)^*} F_{s+1}^{(1,a)^*} F_t^{(1,a)^*}}$$

$$= \frac{F_{n-1}^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_{t-1}^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_{r-1}^{(1,a)^*} F_{s+1}^{(1,a)^*} F_t^{(1,a)^*}} \times \frac{F_n^{(1,a)^*}}{F_{r+1}^{(1,a)^*} F_{s-1}^{(1,a)^*} F_t^{(1,a)^*}} \times \frac{F_{n+1}^{(1,a)^*}}{F_r^{(1,a)^*} F_s^{(1,a)^*} F_{t+1}^{(1,a)^*}}$$

$$= \begin{bmatrix} n-1 \\ r,s,t-1 \end{bmatrix}_{RF} \begin{bmatrix} n \\ r-1,s+1,t \end{bmatrix}_{RF} \begin{bmatrix} n \\ r+1,s-1,t \end{bmatrix}_{RF} \begin{bmatrix} n+1 \\ r,s,t+1 \end{bmatrix}_{RF},$$ as desired.

4.2 Bounds for RF-Trinomial numbers:

In this section, we discuss about the bounds of RF-trinomial numbers in terms of α, where $\alpha = \frac{1+\sqrt{1+4a}}{2}$ is the root of characteristic equation $x^2 - x - a = 0$ of $F_n^{(1,a)}$.

Theorem 4.11: $\alpha^{(t-1)(r+s)+rs} \leq \begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} \leq \alpha^{(t+1)(r+s)+rs}$.

Proof: Since $\alpha^{n-2} \leq F_n^{(1,a)} \leq \alpha^{n-1}$; for all $n \geq 1$, from the definition of RF-trinomial numbers, we have

$$\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF}$$

$$= \frac{F_n^{(1,a)} \times F_{n-1}^{(1,a)} \times \cdots \times F_1^{(1,a)}}{\left(F_r^{(1,a)} \times F_{r-1}^{(1,a)} \times \cdots \times F_1^{(1,a)}\right)\left(F_s^{(1,a)} \times F_{s-1}^{(1,a)} \times \cdots \times F_1^{(1,a)}\right)\left(F_t^{(1,a)} \times F_{t-1}^{(1,a)} \times \cdots \times F_1^{(1,a)}\right)}$$

$$= \frac{F_n^{(1,a)} \times F_{n-1}^{(1,a)} \times \cdots \times F_{n-r-s+1}^{(1,a)}}{\left(F_r^{(1,a)} \times F_{r-1}^{(1,a)} \times \cdots \times F_1^{(1,a)}\right)\left(F_s^{(1,a)} \times F_{s-1}^{(1,a)} \times \cdots \times F_1^{(1,a)}\right)}.$$

Thus, $\frac{\alpha^{(n-2)+(n-3)+\cdots+(n-r-s-1)}}{(\alpha^{(r-1)+(r-2)+\cdots+0})(\alpha^{(s-1)+(s-2)+\cdots+0})} \leq \begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} \leq \frac{\alpha^{(n-1)+(n-2)+\cdots+(n-r-s)}}{(\alpha^{(r-2)+(r-3)+\cdots+(-1)})(\alpha^{(s-2)+(s-3)+\cdots+(-1)})}$.

This now gives $\alpha^{(t-1)(r+s)+rs} \leq \begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} \leq \alpha^{(t+1)(r+s)+rs}$, as required.

This result can be illustrated as follows.

Illustration 4.12: If we consider $n = 7$ and $r = s = 2$ and $t = 3$,

then we have $\begin{bmatrix} n \\ r,s,t \end{bmatrix}_{RF} = \begin{bmatrix} 7 \\ 2,2,3 \end{bmatrix}_{RF} = (1 + 5a + 6a^2 + a^3)(1 + 4a + 3a^2)(1 + 3a + a^2)(1 + 2a)$.

If we take $a = 1$ then we get, $\begin{bmatrix} 7 \\ 2,2,3 \end{bmatrix}_{RF} = (13)(8)(5)(3) = 1560$

Also, $\alpha^{(t-1)(r+s)+rs} = \alpha^{12} = 321.9157$ and $\alpha^{(t+1)(r+s)+rs} = \alpha^{20} = 15120.6460$.

Hence, $\alpha^{12} < \begin{bmatrix} 7 \\ 2,2,3 \end{bmatrix}_F < \alpha^{20}$, as anticipated.

5. Conclusion:

This paper examined the Right-a Double, Right-a Triple Fibonomial numbers, and RF-Trinomial Numbers, presenting their definitions and main properties. Relationships among these forms were established, and several important identities were derived. An equivalent of Gould's Star of David Theorem was also obtained for the triple form, showing interesting arithmetic symmetries. In addition, the concept of RF-Trinomial Numbers was introduced as a three-dimensional extension of the Fibonomial idea. Overall, the study extends the understanding of Fibonomial numbers and highlights their combinatorial structure.

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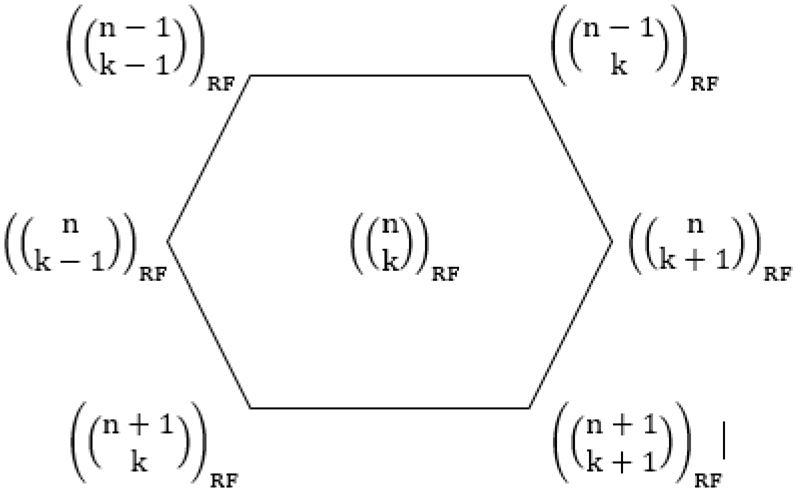

Figure 1: Star of David for Right-a Double Fibonomial numbers

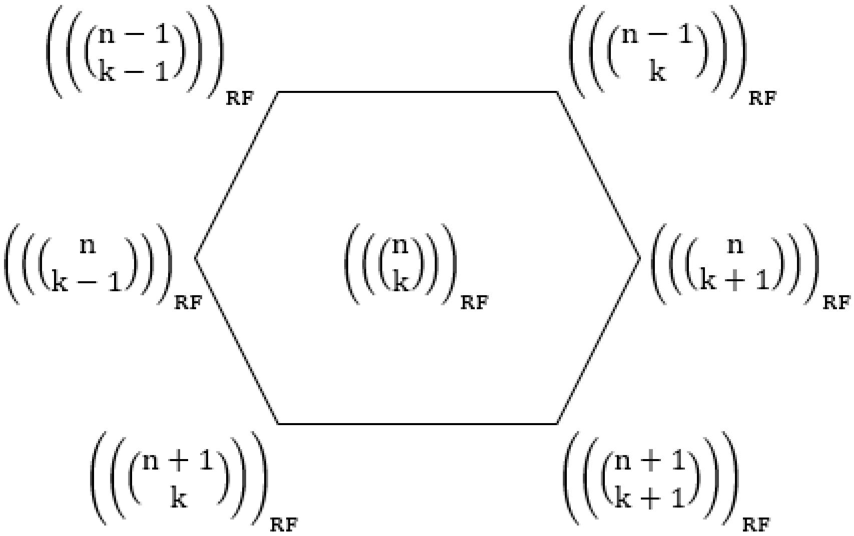

Figure 2: Star of David for Right-a triple Fibonomial numbers